Mathematics Advanced Year 11

Trigonometric Functions Topic Guide

The Mathematics syllabuses are the documents used to inform the scope of content that will be assessed in the HSC examinations.

Topic Guides provide support for the Mathematics Stage 6 courses. They contain information organised under the following headings: Prior learning; Terminology; Use of technology; Background information; General comments; Future study; Considerations and teaching strategies; Suggested applications and exemplar questions.

Topic Guides illustrate ways to explore syllabus-related content and consequently do not define the scope of problems or learning experiences that students may encounter through their study of a topic. The terminology list contains terms that may be used in the teaching and learning of the topic. The list is not exhaustive and is provided simply to aid discussion.

Please provide any feedback to the Mathematics and Numeracy Curriculum Inspector.

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# Topic focus

The topic Trigonometric Functions involves the study of periodic functions in geometric, algebraic, numerical and graphical representations.

A knowledge of trigonometric functions enables the solving of practical problems involving triangles or periodic graphs, such as waves and signals.

The study of trigonometric functions is important in developing students’ understanding of periodic behaviour, a property not possessed by any previously studied functions. Using this property, mathematical models have been developed that describe the behaviour of many naturally occurring periodic phenomena, such as vibrations or waves, as well as oscillatory behaviour found in pendulums, electric currents and radio signals.

# Prior learning

The material in this topic builds on content from the Measurement and Geometry Strand and Number and Algebra Strand of the *Mathematics K–10 Syllabus*, including the 5.2 substrand of Right-Angled Triangles and the 5.3 substrands of Trigonometry and Pythagoras’ Theorem, Algebraic Techniques, Equations and Properties of Geometrical Figures.

# Terminology

|  |  |  |
| --- | --- | --- |
| ambiguous case   angles of any magnitude  angle of depression  angle of elevation  arc length  circular measure  compass bearing  complementary angle results | cosecant  cotangent  cosine rule  exact value  geometric construction  identity ****  periodicity  Pythagorean identities | radian  reciprocal trigonometric functions  related angle secant  sine rule  subtend  true bearing  unit circle |

# Use of technology

Dynamic geometry software could be used to investigate the relationships between sides and angles in triangles, and to provide a visual introduction to the sine rule. Additionally such software could be used to illustrate three-dimensional problems to encourage students to view a three-dimensional construction from different perspectives.

Locations and distances between points can be obtained from various interactive maps available on the internet. This data could be used as a source of measurements leading to the calculation of sides and angles of triangles.

Graphs of and are easily developed from the definitions based on the unit circle and should be illustrated using appropriate software applications.

# Background information

Almost all early civilisations used the shadow cast by a vertically positioned stick to observe the motion of the Sun and tell time. This instrument is now called a Gnomon, the Greek name of an L-shaped instrument used to draw a right angle. Tables of the numerical sequences of Gnomon shadow lengths and correlating times of day are often viewed as ancestors to the tangent and cotangent.

The foundations of trigonometry were laid as early Babylonian, Greek, Hellenistic, Indian, and Arabic mathematicians investigated astronomical problems using numerical and geometric techniques, specifically the geometry of the circle. Hipparchus of Nicaea (c.190–120BC) is often referred to as the ‘father of trigonometry’ because the earliest evidence of trigonometric tables relating to chord lengths was attributed to him. As an astronomer, Hipparchus, focused mainly on spherical triangles, such as those formed between three stars.

Trigonometry was established as a distinct branch of mathematics during the 12th and 13th centuries. However it was not until the 16th century that trigonometry moved from being predominantly geometric to an algebraic-analytic discipline.

Students may be interested in researching the history of the terms used in trigonometry, and this can assist in understanding certain concepts. For example, terms such as sine and cosine evolved within the context of astronomy and spherical geometry, whereas tangent and cotangent from shadows and the Gnomon. The first trigonometric ratios to be calculated were related to the sine ratio. Early expressions for cosine included ‘sinus complementi’ and ‘cosinus’. As these expressions indicate, the cosine of an angle is the sine of the complement of that angle, so that once a set of values has been established for the sine ratio, these can be used for the cosine ratio as well. For example, the cosine of is the sine of the complement of and is therefore the same value as the sine of .

# General comments

This topic prepares students for the study of the trigonometric functions, which are important in many practical applications and essential for many more advanced aspects of mathematics.

Students are required to develop a strong conceptual understanding of the trigonometric ratios of angles of any magnitude.

Exact values of trigonometric ratios are used in theoretical work and in technical calculations where their use avoids rounding errors.

# Future study

Students need to ensure that they can efficiently manipulate trigonometric expressions, model practical and abstract scenarios involving trigonometry and solve problems that involve trigonometric applications to facilitate work in later topics.

For students studying the Mathematics Extension 1 course, this topic can be combined with or lead straight into the subtopic ME-T2: Further Trigonometric Identities.

# Subtopics

* MA-T1: Trigonometry and Measure of Angles Paperclip icon
* MA-T2: Trigonometric Functions and Identities

## MA-T1: Trigonometry and Measure of Angles Paperclip icon

### Subtopic focus

The principal focus of this subtopic is to solve problems involving triangles using trigonometry, and to understand and use angular measure expressed in radians and degrees. This has practical and analytical applications in areas including surveying, navigation, meteorology, architecture, construction and electronics.

Students develop techniques to solve problems involving triangles, and then extend these ideas to include the exact ratios for angles, and also to the study of non-right-angled triangles. This introduces the need to define the trigonometric ratios for obtuse angles, which is followed by the establishment of trigonometric ratios of angles of any size. Radians are introduced as another measure in which angles of any size can be found. Radians are important for the study of the calculus of trigonometric functions in Year 12.

Within this subtopic, schools have the opportunity to identify areas of Stage 5 content which may need to be reviewed to meet the needs of students.

## T1.1 Trigonometry

### Considerations and teaching strategies

* Review of the following may be needed to meet the needs of students:
  + angle measures, representations and conversions
  + Pythagoras’ theorem and the trigonometric ratios for angles in right-angled triangles
  + angles of elevation and depression
  + true bearings (three-figure bearings, eg , )
  + compass bearings expressed as:
    - one of the 16 points of a mariner’s compass, eg SSW
    - the number of degrees east or west of the north-south line, eg NE, SW
    - common descriptions, eg ‘due East’, ‘South-West’.
* The sine rule, cosine rule, and area rule ():
  + Proofs of the sine rule, cosine rule, and area rule should be demonstrated to students. They will not be expected, however, to reproduce these proofs.
  + Explore the complementary and supplementary angle relationships:

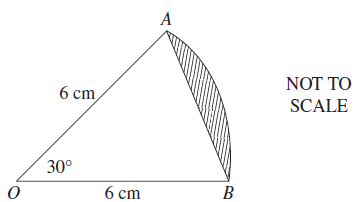
and , where is an acute angle

, , where is an acute angle.

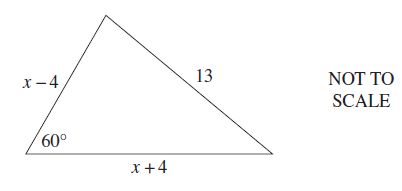
* + The ambiguous case of the sine rule should be demonstrated geometrically and determined algebraically to show how two possible triangles may exist.
  + Pythagoras’ theorem is a special case of the cosine rule.
  + Students should be familiar with various forms of the sine and cosine rules and how to change the subject of the formula.   
    eg: or   
    and or
* Students could either be given a diagram for a three-dimensional problem or construct a diagram from information provided.

### Suggested applications and exemplar questions

* In the diagram, is a sector of the circle with centre and radius 6 cm, where . Determine the exact value of the area of the triangle .

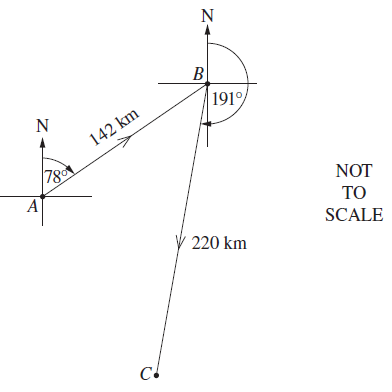


* Find the value of in the following diagram.

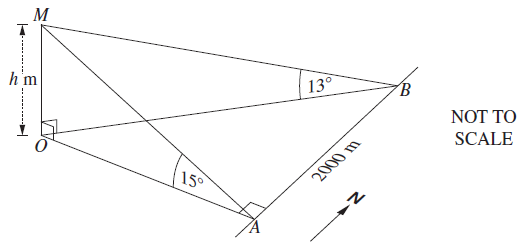


* Chris leaves island in a boat and sails 142 km on a bearing of to island . Chris then sails on a bearing of for 220 km to island , as shown in the diagram below.

1. Show that the distance from island to island is approximately 210 km.
2. Chris wants to sail from island directly to island . On what bearing should Chris sail? Give your answer correct to the nearest degree.



* Determine the possible dimensions for triangle given cm, and cm.
* A person walks 2000 metres due north along a road from point to point . The point is due east of a mountain , where is the top of the mountain. The point is directly below point and is on the same horizontal plane as the road. The height of the mountain above point is metres. From point , the angle of elevation to the top of the mountain is . From point , the angle of elevation to the top of the mountain is . Determine the height of the mountain (diagram on next page).



* The Eiffel Tower is located in Paris, a city built on a flat floodplain. Three tourists , and are observing the Eiffel Tower from the ground. is due north of the tower, is due east of the tower, and is on the line-of-sight from to and between them. The angles of elevation to the top of the Eiffel Tower from , and are , and , respectively. Determine the bearing of from the Eiffel Tower.

## T1.2 Radians

### Considerations and teaching strategies

* Angles of any magnitude should be illustrated with reference to the unit circle:   
   .
* , and should be defined for any angle with relation to the unit circle.
  + The term ‘related angle’ should be introduced.
  + Graphs should be drawn showing these ratios as functions of the angular measure in degrees.
  + Trigonometric ratios for angles which can be written in the form , and
  + , where should be obtained.
  + Explore and use the results for reflex angle relationships where :

, ,

, , .

* + The trigonometric ratios for °, °, °, °, and ° should be established from the definitions.
  + Angles greater than occur when making more than one complete revolution.
* Recall aids related to which quadrants of the unit circle contain positive results for and could be used such as CAST or mnemonics like *All Stations To Central* (ASTC).
* For a given angle, the ratio of the length of the arc it subtends to the radius of the arc provides the natural measure of angles, and is called circular measure (or radian measure). This leads to the relationship
  + When the arc is a semicircle, , and so . This gives the conversion that an angle of has a circular measure of .
  + It should be noted that a ‘radian’ is not a unit of measure. It has no dimensions. However, the term is often used to indicate that an angle is given in circular measure and not in degrees. For example, means the sine of an angle whose circular measure is , and this may be read as ‘ radians’ to distinguish it from , the sine of an angle of 3 degrees.
  + Practice should be given so that exact equivalents are known for common angle sizes and so that accuracy is developed in approximating sizes given in one measure by sizes in another.
  + The formula , for the length of an arc subtending an angle at the centre of a circle of radius , and the formula for the area of the corresponding sector should be derived.
* Ratios for and radians should be known as exact values.
* Using circular measure, sine and cosine are now defined as functions of a real variable: for each real number , is defined as the sine of an angle whose circular measure is , is defined as the cosine of this angle.
  + The functions are defined for all real.Their graphs should be drawn using computer software, graphing calculators, or by-hand methods.
  + The function may then be defined in terms of and , the domain of definition found, and the graph drawn.
* The previous work on angles of any magnitude and the unit circle should be revisited using radians in place of degrees.
* Consider the solution of a problem involving the ambiguous case of the sine rule, and use the unit circle and angles of any magnitude to appreciate that this is an example of the use of the related angle , where and .
* Once familiarity with the trigonometric ratios of angles of any magnitude is attained, some practice in solving simple equations, of the type likely to occur in later applications, should be discussed. Solutions should be given in exact form where possible. Examples of equations to be solved for or include:

(a) (b) .

### Suggested applications and exemplar questions

* Find the exact values of:

1. (b) (c)

* Convert radians to degrees.
* Find the exact value of .
* Solve:

1. for
2. for

* Consider a circular clock-face of radius 1 unit centered on the origin. The coordinates of the ‘twelve o’clock’ position are (0,1) and the coordinates of the ‘one o’clock’ position are .
  + What are the coordinates of the other hour positions?
  + Identify the value of in each case by writing each pair of coordinates in the form .

Note that this can be used to reinforce the exact values of the trigonometric ratios for angles such as , , , , , …

* Find the perimeter and the area of the segment cut off by a chord of length 8 cm in a circle centre and radius 6 cm. Give your answers correct to 3 significant figures.
* A chord of a circle which subtends an angle of at the centre of the circle cuts off a segment equal in area to of the area of the whole circle.

1. Show that .
2. Verify that radians, correct to 2 decimal places.

## MA-T2: Trigonometric functions and identities

### Subtopic focus

The principal focus of this subtopic is to use trigonometric identities and reciprocal relationships to simplify expressions, to prove equivalences and to solve equations.

Students develop their ability to prove identities, simplify expressions and solve trigonometric equations. Trigonometric expressions and equations provide a powerful tool for modelling quantities that vary in a cyclical way such as tides, seasons, demand for resources, and alternating current. The solution of trigonometric equations may require the use of trigonometric identities.

### Considerations and teaching strategies

* The notation to represent needs to be explained to students.
* In modern usage, csc is the abbreviation for ‘cosecant’.
* When sketching the graphs for the reciprocal trigonometric functions, identify properties of those graphs such as asymptotes, periodicity, maximum and minimum values and their relationship to the graphs of the trigonometric functions, noting that maximum and minimum values are local not global for and .

### Suggested applications and exemplar questions

* Find exact values of: (a) (b) (c)
* On the same set of axes, sketch and for . Using graphing technologies or otherwise, find the values of for which .
* Show that:

1. (b)

* If , express in terms of : .
* Solve for
* (a) Prove that .

(b) Hence prove that .

* Given that , and that . Find the values of:

1. (b) .

* Express in terms of and hence solve the equation

for .